NASA TT F-9872

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Acta Physica Sinica, Vol. 20, No. 9, pp 873-889, 1964

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON D.C. FEBRUARY 1966

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ABSTRACT

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This article studies the properties of a superconducting film in a magnetic field close to the critical temperature T_c. The magnetic field is parallel to the film surface.

This article discusses the theory of Gor'kov (Ref. 3), and shows that this theory employs the local approximation which was introduced in his expression for the Green's function of an electron in a magnetic field $\tilde{G}^{o}_{\omega}(rr')$. The authors of this article assume that this expression for $\tilde{G}^{o}_{\omega}(rr')$ is not applicable to fine films, and it is replaced by expansion in a perturbation theory series in powers of the vector potential.

The compensation equation and Gor'kov's current equation are generalized to the case of the film. The equations are solved for a film having the thickness $d < \delta_o(T)$, and the expressions are obtained for the vector potential, the magnetic moment, energy gap, and the critical current. The phase conversion of the film in a magnetic field is studied, and the

*Received September 9, 1963.

expression for the critical field is obtained.

The results obtained coincide with the Ginzburg-Landau theory in the case of a fixed film ($d \gg \xi_o$) and differ considerably from this theory in the case of a fine film ($d < \xi_o$) (in addition to the expression for the critical current and the criterion for a type of phase conversion of a superconducting film in a magnetic field). The difference is clearly expressed in the dependence of all quantities on the film thickness. The results obtained are valid in the case $\delta_o(T) > d \gg d^*$, where $2d^* = \left(\frac{1}{0.36} \frac{\xi_o}{p_o}\right)^{1/2}$ (for Sn, $2d^* \sim 10^{-6}$ cm) for the approximation assumed in this article. Author

I. INTRODUCTION

In a previous paper published not long ago, one of the authors /873*
used the method of Green's function and extended the theory of BCS (Ref 8) to the case of superconducting metal films, and provided a theory of superconductivity of thin metal films. It discussed the equilibrium properties of superconducting thin films, and proved that when the thickness of a superconductive film is thin enough, its critical temperature T_c, the energy gap, and thermodynamic properties change periodically with the film thickness. In this paper and a later one, we shall study the properties of a superconducting film in a magnetic field near the critical temperature T_c. The intensity of the magnetic field is arbitrary and is not limited to the

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*Note: Numbers in the margin indicate pagination in the original foreign text.

case of a very small critical magnetic field.

As is well known, the property of a superconducting substance under the above conditions can be discussed by the abstract of Ginzburg-Landau theory (Ref. 2) (abbreviated as the GL theory). For an infinite superconductor, Gor'kov (Ref. 3) has proved that the basic equations (i.e., the generally called Ginzburg-Landau equations) in the GL theory can be derived from the microscopic theory of superconduction. These equations are equivalent to the compensation equation and the current equation in the microscopic theory.

A superconducting thin film has an extraordinary special feature. When its thickness 2d is so small that it can be compared with the coherence length ξ_0, the relationship between the current and the vector potential is not local even in the range near T_c. This represents an important difference between a superconducting thin film and an infinite superconductor.

In this paper we shall analyze Gor'kov's theory (the third section), and shall point out that in the process of deriving the GL equation, Gor'kov actually used a local approximation. This approximation is mainly /874 introduced implicitly in the approximate solution of the Green's function ($\tilde{G}^0_\omega$ (r, r')); ($\tilde{G}^0_\omega$ (r, r')) is the Green's function of a normal electron in a magnetic field. For this reason, Gor'kov's theory is only applicable to a superconducting film of large enough thickness. In order to extend Gor'kov's theory to a superconducting film of smaller thickness, the crucial problem lies in considering the non-local characteristics of a superconducting thin film; therefore, we must discard the local approximation used by Gor'kov. In the third section, we shall show that the approximate expression of Green's function $\tilde{G}^0_\omega(r,r')$ used by Gor'kov must be discarded. Instead, the small perturbation expansion of $\tilde{G}^0_\omega(r,r')$ in power series of the vector potential can be used.

Applying the method given in the literature (Ref. 1), one can readily extend the basic equations in Gor'kov's theory to the case of a superconducting film. In Section 2, we give the compensation equation and the current equation applicable for the range near the critical temperature T_c. These equations are expressed through $\tilde{G}^0_\omega(r,r')$, and are valid for films of any thickness. When the film thickness is smaller than $2\delta_0(T)$ [$\delta_0(T)$ represents the weak field penetration depth of the infinite conductor], the distribution of the magnetic field in a metal film is not very different between the superconducting phase and the normal phase. Using these physical characteristics and under the assumption of constant energy gap, the solutions of the compensation equation and the current equation are obtained. Also, the magnetic moment, energy gap, and the critical magnetic field (the 417th section) of the superconducting thin film are calculated. In the eighth section, the critical current is calculated.

The results we obtained (except the critical current and the criterion for the order of the phase transition) are considerably different from the GL theory when $d < \xi_0$. The difference between the two is expressed particularly well in the dependence relation on d; only when $d \gg \xi_0$ does the result of this paper agree with that of the GL theory.

In another article (Ref. 4), we further investigated the second-order phase transition critical magnetic field and compared it with experiment. The theory and the experimental results agree very well.

The results of this paper and (Ref. 4) are applicable to superconducting films which satisfy the condition $\delta_0(T) > d \gg d^*$, where 2d is the film thickness, $2d^* = \left(\frac{1}{0.36}\frac{\xi_0}{p_0}\right)^{1/2}$, and p_0 is the wave vector of electrons around the Fermi surface of normal metals. For Sn, it can be estimated that $2d^* \sim 10^{-6}$ cm. As to the properties of superconducting thin film with $d \lesssim d^*$, we shall discuss them in another article, where those which are neglected in section 417 are going to be important.

II. BASIC EQUATIONS

Consider an infinite superconducting film with thickness 2d. Assume that the magnetic field is parallel to the film surface along the r_3 direction, and the current passes along the r_2 direction (See Figure 1). The magnetic field and the current are both functions of r_1 alone. Therefore, we can select the vector potential which has only a r_2 direction component and besides is a function of r_1 only: $\mathbf{A} = (0, A(r_1), 0)$. At this time, the magnetic field $H = \frac{\partial A(r_i)}{\partial r_i}$.

According to the method given in (Ref. 1), it is not difficult to obtain the equation satisfied by the Green's function of the superconducting film in the magnetic field. If we make the selection postulated by our criterion, the energy gaps are real, and are all functions of r_1 alone. Therefore, these equations can be written as:

$$\left\{i\omega + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_1^2} + \frac{\hbar^2}{2m}\left(\frac{\partial}{\partial r_2} - \frac{ie}{\hbar c}A(r_1)\right)^2 + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_3^2} - U(\mathbf{r}) + \mu\right\} G_\omega(\mathbf{r}, \mathbf{r}') + \Delta(r_1)F_\omega^+(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{1a}$$

$$\left\{-i\omega + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_1^2} + \frac{\hbar^2}{2m}\left(\frac{\partial}{\partial r_2} + \frac{ie}{\hbar c}A(r_1)\right)^2 + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_3^2} - U(\mathbf{r}) + \mu\right\} F_\omega^+(\mathbf{r}, \mathbf{r}') + \Delta(r_1)G_\omega(\mathbf{r}, \mathbf{r}') = 0, \tag{1b}$$

where the energy gap $\Delta(r_1)$ is /875

$$\Delta(r_1) = |g|kT\sum_\omega F_\omega^+(\mathbf{r}, \mathbf{r}'), \tag{2}$$

U(r) represents the potential energy of the metal film boundary. We have selected the directional very well. Green's functions of the superconducting film appearing in equation (1) and those introduced in the following sections all satisfy the appropriate boundary conditions. We let $G_\omega(r,r')$ and $F^+_\omega(r, r')$, etc., satisfy, in the direction of the film thickness, the boundary condition of that partial derivative equal to zero. For example, the condition for $G_\omega(r,r')$ in the thickness direction is

$$\left.\frac{\partial G_\omega(\mathbf{r},\mathbf{r}')}{\partial r_1}\right|_{r_1=0\ \mathrm{or}\ 2d} = \left.\frac{\partial G_\omega(\mathbf{r},\mathbf{r}')}{\partial r_1'}\right|_{r_1'=0\ \mathrm{or}\ 2d} = 0. \tag{3}$$

From the current formula obtained below, it is not difficult to see that the physical meaning of the boundary condition lies in the fact that the current along the normal direction of the boundary is zero. The Green's function satisfying the above condition can be expanded, using the solution of the single-particle Schrödinger equation without an external field, and satisfying the same boundary condition. The solution is

$$\psi_{kpq}(r_1, r_2, r_3) = \varphi_k(r_1)\left(\frac{1}{2\pi}\right)e^{i(pr_2+qr_3)},$$

$$\varphi_k(r_1) = \begin{cases}\sqrt{\dfrac{1}{2d}} & \text{when } k = 0\\[2ex] \sqrt{\dfrac{1}{d}}\cos kr_1 & \text{when } k \neq 0\end{cases}$$

$$\varepsilon_{kpq} = \frac{\hbar^2}{2m}(k^2 + p^2 + q^2);$$

where $K = \frac{n\pi}{2d}$, $n = 0, 1, 2, \ldots$. This kind of expansion represents an extension, to the case of superconducting films, of the expansion method from plain waves, which is generally used in an infinite sample.

Using the method of (Ref. 3), we can introduce the Green's function $\tilde{G}^0_\omega(r, r')$ of normal electrons in a magnetic field. It satisfies the equation

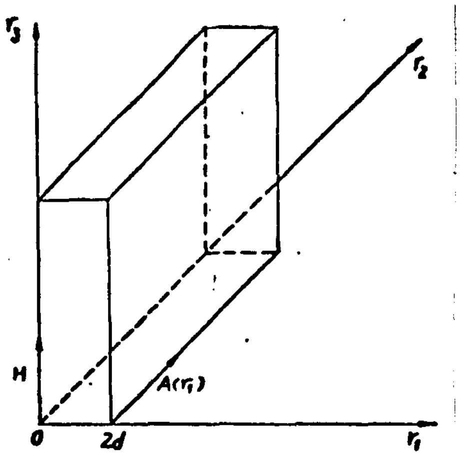

Figure 1

A Superconducting Film of Thickness 2d

(The Magnetic Field is Plotted Along the r_3 Direction, and the Vector Potential is Plotted along the r_2 Direction)

$$\left\{i\omega + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_1^2} + \frac{\hbar^2}{2m}\left(\frac{\partial}{\partial r_2} - \frac{ie}{\hbar c}A(r_1)\right)^2 + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_3} - U(\mathbf{r}) + \mu\right\}\tilde{G}^0_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'). \quad (4)$$

Applying this, we can transform (1) into a form of integral equations:

$$G_\omega(\mathbf{r}, \mathbf{r}') = \tilde{G}^0_\omega(\mathbf{r}, \mathbf{r}') - \int d\mathbf{l}\,\tilde{G}^0_\omega(\mathbf{r}, \mathbf{l})\Delta(l_1)F^+_\omega(\mathbf{l}, \mathbf{r}'), \quad (5\ a)$$

$$F^+_\omega(\mathbf{r}, \mathbf{r}') = \int d\mathbf{l}\,\tilde{G}^0_{-\omega}(\mathbf{l}, \mathbf{r})\Delta(l_1)G_\omega(\mathbf{l}, \mathbf{r}'). \quad (5\ b)$$

When $\Delta t = 1 \frac{T}{T_c} << 1$, equation (5) can be expanded with respect to $\Delta(r_1)$ by the small perturbation theory. Under the approximation accurate to the Δ^2 term, $G_\omega(r, r')$ can be expressed as /876

$$G_\omega(\mathbf{r}, \mathbf{r}') = \tilde{G}^0_\omega(\mathbf{r}, \mathbf{r}') - \iint d\mathbf{l}\,d\mathbf{m}\,\tilde{G}^0_\omega(\mathbf{r}, \mathbf{l})\Delta(l_1)\tilde{G}^0_\omega(\mathbf{m}, \mathbf{r}')\Delta(m_1)\tilde{G}^0_{-\omega}(\mathbf{m}, \mathbf{l}). \quad (6)$$

Substituting (6) into (5b), and using equation (2), we obtain the gap equation applicable near T_c.

$$\begin{aligned}\Delta(r_1) = {} & |g|kT\sum_\omega\int d\mathbf{l}\,\tilde{G}^0_\omega(\mathbf{r}, \mathbf{l})\Delta(l_1)\tilde{G}^0_{-\omega}(\mathbf{l}, \mathbf{r}) - \\ & - |g|kT\sum_\omega\iiint d\mathbf{l}\,d\mathbf{m}\,d\mathbf{s}\,\tilde{G}^0_\omega(\mathbf{l}, \mathbf{m})\Delta(m_1)\tilde{G}^0_\omega(\mathbf{s}, \mathbf{r})\Delta(s_1)\tilde{G}^0_{-\omega}(\mathbf{s}, \mathbf{m})\Delta(l_1) \times \\ & \times \tilde{G}^0_{-\omega}(\mathbf{l}, \mathbf{r}).\end{aligned} \quad (7)$$

In the same way, the current equation near T_c is obtained

$$j(r_1) = -\frac{ie\hbar}{m}\left(\frac{\partial}{\partial r_2'} - \frac{\partial}{\partial r_2}\right)_{r_2'\to r_2} kT \sum_\omega \iint dl\,dm\,\tilde{G}_\omega^0(\mathbf{r}, \mathbf{l})\Delta(l_1)\tilde{G}_\omega^0(\mathbf{m}, \mathbf{r}') \times \times \Delta(m_1)\tilde{G}_{-\omega}^0(\mathbf{m}, \mathbf{l}). \tag{8}$$

The compensation equation (7) and the current equation (8) represent an extension of the corresponding equations in Gor'kov's theory. The integration, with respect to the coordinate variables, which appears in equations (5)-(8) and in the equations of the following sections, is limited to the interior of the metal film.

III. LOCAL APPROXIMATION IN GOR'KOV'S THEORY

$\tilde{G}_\omega^0(r, r')$ introduced in the previous section is defined by equation (4), and it satisfies a boundary condition similar to equation (3). Gor'kov gave an approximate expression for $\tilde{G}_\omega^0(r, r')$ as, (Ref. 3)

$$\tilde{G}_\omega^0(\mathbf{r}, \mathbf{r}') = e^{\frac{ie}{\hbar c}A(r_2)(r_2 - r_2')} G_\omega^0(\mathbf{r}, \mathbf{r}'), \tag{9}$$

where $G_\omega^0(r, r')$ is the Green's function of a free particle. With a metal film, it satisfies the equation

$$\left\{i\omega + \frac{\hbar^2}{2m}\nabla^2 - U(\mathbf{r}) + \mu\right\} G_\omega^0(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{10}$$

Therefore, we have

$$G_\omega^0(\mathbf{r}, \mathbf{r}') = \iint \frac{dp\,dq}{(2\pi)^2} \sum_k \frac{1}{i\omega - \xi_{kpq}} \varphi_k(r_1)\varphi_k(r_1') e^{ip(r_2 - r_2') + iq(r_3 - r_3')}, \tag{11}$$

where $\xi_{kpq} = \varepsilon_{kpq} - \mu$. In the case of an infinite super conductor, if a solution of the equation (9) form is substituted into equations similar to (7) and (8), if the energy gap $\Delta(r_1)$ is expanded in Taylor series with respect to r_1, if terms up to $\frac{\partial^2\Delta(r_1)}{\partial r_1^2}$ are retained, and if the exponent of equation (9) is expanded retaining terms up to the square of the vector potential, the GL equation is obtained. This is exactly what Gor'kov did.

For superconducting films, if we follow the same procedure as Gor'kov's, it is not hard to prove that the results obtained from equations (7) and (1), under the approximation of constant energy gap, are exactly the same as those in the GL theory with k = 0. There is a difference between the two only when corrections are introduced, considering the dependence of the energy gap on r_1. When the film thickness is small enough, this difference is not important at all. This, however, does not imply that the GL theory is accurate even in the case of a superconducting thin film. The problem consists of whether it is reasonable to apply Gor'kov's approximate solution here.

In order to analyze this problem, we should re-examine the solution of $\tilde{G}^0_\omega(r,r')$. Let us write the differential equation (4) which $\tilde{G}^0_\omega(r,r')$ satisfy as /877

$$\left\{i\omega + \frac{\hbar^2}{2m}\nabla^2 - U(\mathbf{r}) + \mu\right\}\tilde{G}^0_\omega(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') + + \left[\frac{ie\hbar}{mc}A(r_1)\frac{\partial}{\partial r_2} + \frac{e^2}{2mc^2}A^2(r_1)\right]\tilde{G}^0_\omega(\mathbf{r},\mathbf{r}'). \tag{4}$$

We would like to point out that Gor'kov's approximate solution (9) can be obtained in the following manner. First, let us neglect the term containing A^2 on the right-hand side of equation (4) (this approximation is called approximate A). Then, using equation (10), we can write it as an integral equation:

$$\tilde{G}^0_\omega(\mathbf{r},\mathbf{r}') = G^0_\omega(\mathbf{r},\mathbf{r}') + \int G^0_\omega(\mathbf{r},\mathbf{l})\frac{ie\hbar}{mc}A(l_1)\frac{\partial}{\partial l_2}\tilde{G}^0_\omega(\mathbf{l},\mathbf{r}')d\mathbf{l}.$$

Proceeding with the substitution, an infinite series is obtained:

$$\tilde{G}^0_\omega(\mathbf{r},\mathbf{r}') = G^0_\omega(\mathbf{r},\mathbf{r}') + \int d\mathbf{l} G^0_\omega(\mathbf{r},\mathbf{l}) \frac{ie\hbar}{mc} A(l_1) \frac{\partial}{\partial l_2} G^0_\omega(\mathbf{l},\mathbf{r}') + + \iint d\mathbf{l} d\mathbf{m} G^0_\omega(\mathbf{r},\mathbf{l}) \frac{ie\hbar}{mc} A(l_1) \frac{\partial}{\partial l_2} G^0_\omega(\mathbf{l},\mathbf{m}) \frac{ie\hbar}{mc} A(m_1) \frac{\partial}{\partial m_2} G^0_\omega(\mathbf{m},\mathbf{r}') + \cdots. \tag{12}$$

Then let us substitute the vector potential under each integral sign in (12) with $A(r_1)$ (we call this approximation B), and we obtain

$$\tilde{G}^0_\omega(\mathbf{r},\mathbf{r}') = G^0_\omega(\mathbf{r},\mathbf{r}') + \frac{ie\hbar}{mc} A(r_1) \int d\mathbf{l} G^0_\omega(\mathbf{r},\mathbf{l}) \frac{\partial}{\partial l_2} G^0_\omega(\mathbf{l},\mathbf{r}') + + \left(\frac{ie\hbar}{mc}\right)^2 A^2(r_1) \iint d\mathbf{l} d\mathbf{m} G^0_\omega(\mathbf{r},\mathbf{l}) \frac{\partial}{\partial l_2} G^0_\omega(\mathbf{l},\mathbf{m}) \frac{\partial}{\partial m_2} G^0_\omega(\mathbf{m},\mathbf{r}') + \cdots. \tag{13}$$

It is not difficult to prove that the nth term of (13) equals

$$\frac{1}{(n-1)!}\left(\frac{ie}{\hbar c}\right)^{n-1} A^{n-1}(r_1)(r_2 - r_2')^{n-1} G^0_\omega(\mathbf{r},\mathbf{r}'). \tag{14}$$

Taking n = 2 as an example, and using equation (11), the n = 2 term can be expressed as

$$\frac{ie\hbar}{mc} A(r_1) \iint \frac{dp\,dq}{(2\pi)^2} \sum_k \frac{ip}{(i\omega - \xi)^2} \varphi_k(r_1)\varphi_k(r_1') e^{ip(r_2 - r_2') + iq(\cdot - r_3')},$$

Using the transformation $\dfrac{ip}{(i\omega-\xi)^2} = i\dfrac{m}{\hbar^2}\dfrac{\partial}{\partial p}\left(\dfrac{1}{i\omega-\xi}\right)$,

and integrating by parts, we obtain

$$\frac{ie}{\hbar c} A(r_1)(r_2 - r_2') \iint \frac{dp\,dq}{(2\pi)^2} \sum_k \frac{1}{i\omega - \xi} \varphi_k(r_1)\varphi_k(r_1') e^{ip(r_2 - r_2') + iq(r_3 - r_3')}.$$

that is,

$$\frac{ie}{\hbar c} A(r_1)(r_2 - r_2') G^0_\omega(\mathbf{r},\mathbf{r}').$$

In general, equation (14) can be proved in the same manner. From equation (14), equation (13) can be seen to correspond to equation (9).

The above derivation clearly explains the fact that, in the Gor'kov approximation, two approximations are actually involved, i.e., approximation A and approximation B mentioned above.

In the introduction, we have pointed out that a theory used to study the behavior of a superconducting film in a magnetic field must take into consideration the non-local feature of the current vs. vector potential relationship. Essentially speaking, approximation (b) is a local approximation. Under this approximation, equation (8) is a local relationship /878 between the current and the vector potential. Therefore, for the problem we are studying, approximation (B) is not applicable. This can be explained further from another angle. In fact, it is very easy to prove that in the vicinity of T_c, the Green's function $G^0_\omega(r,r')$ of a free particle decreases very rapidly when $|r - r'| > \xi_0$. Since the variation range of the vector potential $A(r_1)$ is the film thickness, for a thick enough film ($2d >> \xi_0$), the relative change of the vector potential within ξ_0 is very slow, and the local approximation (B) is applicable. However, for a relatively thin film ($2d << \xi_0$), the relative change of the vector potential within ξ_0 is very fast, and local approximation (B) is not allowed.

As to approximation (A), it is applicable to any kind of film. In fact the ratio of the square term of $A(r_1)$ to the linear term on the right-hand side of equation (4') can be estimated as $\frac{eA}{cp_p} \lesssim \frac{cH_0\delta_0(T)}{cp0} << 1$. Therefore, it is reasonable to neglect the term containing $A^2(r_1)$ in (4').

Summarizing the above discussion, we reach the following conclusion. Gor'kov's approximate solution (9) which contains a local approximation is not applicable to the problem studied in this paper. However, the expansion formula (12) of $\tilde{G}^0_\omega(r,r')$ which contains approximation (A) only is completely applicable. In the vicinity of T_c, it is sufficient to take only several limited terms. This point is used in all of the following sections. We would like to point out that the approximate expression of $\tilde{G}^0_\omega(r,r')$ thus obtained satisfies the boundary condition (3).

IV. THE VECTOR POTENTIAL AND THE MAGNETIC FIELD

If (12) is substituted into (7) and (8), and a method similar to that of (Ref. 3) is used - that is, the energy gap function under the integral sign of (7) is expanded into Taylor series with respect to r_1, up to the $\frac{\partial^2\Delta(r_1)}{\partial r_1^2}$ term, and terms up to the square term of the vector potential are retained - we would obtain an integro-differential equation. This is an extension of the GL theory to the case of thin films. This equation, however, is very complicated and it is not easy to obtain an analytical result.

In this and later papers, we restrict ourselves to the study of the properties of superconducting films with $d < \delta_0(T)$. From the discussion of the third section and (Ref. 4), we found that the difference between the GL theory and the experiment mainly occurs in superconducting films with $2d \leq \xi_0$. A study of the properties of a film with $2d \leq \xi_0$ is the main objective of this paper. We would like to note that when the temperature is close enough to T_c, $\delta_0(T)$ will be greater than ξ_0. Therefore, in the vicinity of T_c, it is suitable to study the superconducting film subject to the limitation of $d < \delta_0(T)$. For the superconducting film with $d < \delta_0(T)$, under the approximation of constant energy gap, equations (7) and (8) can be solved directly by the following argument. Actually, when $d < \delta_0(T)$, the distribution of magnetic field $H(r_1)$ inside the metal film at the superconducting phase is not much different from that at the normal phase. For this reason, we can express the vector potential $A(r_1)$ with the vector potential of a normal-phase thin film $H_0r_1 + C$ plus a correction term for superconduction, i.e.,

$$A(r_1) = H_0r_1 + C + \tilde{A}(r_1), \tag{15}$$

where H_0 expresses the intensity of the applied magnetic field; C is a constant. Since at $r_1 = 0$ or 2d, $\frac{dA(r_1)}{dr_1} = H_0$, therefore, $\tilde{A}(r_1)$ satisfies the boundary condition

$$\left.\frac{d\tilde{A}(r_1)}{dr_1}\right|_{r_1=0 \text{ и } 2d} = 0. \tag{16}$$

It is apparent that, when $d < \delta_0(T)$, $\tilde{A}(r_1)$ is a small quantity. Substituting (12) into (8), and retaining up to the linear term of $A(r_1)$, the current equation, under the approximation of constant energy gap, can be written as /879

$$\begin{aligned}
j(r_1) = &-\frac{ie\hbar\Delta^2}{m} kT \sum_{\omega} \int \cdots \int d\mathbf{l} d\mathbf{s} d\mathbf{m} \left(\frac{\partial}{\partial r_1'} - \frac{\partial}{\partial r_1}\right)_{r_1' \to r_1} G_{\omega}^0(\mathbf{r}, \mathbf{s}) \times \\
&\times \frac{ie\hbar}{mc} A(s_1) \frac{\partial}{\partial s_2} G_{\omega}^0(\mathbf{s}, \mathbf{l}) G_{\omega}^0(\mathbf{m}, \mathbf{r}') G_{-\omega}^0(\mathbf{m}, \mathbf{l}) - \\
&-\frac{ie\hbar\Delta^2}{m} kT \sum_{\omega} \int \cdots \int d\mathbf{l} d\mathbf{s} d\mathbf{m} \left(\frac{\partial}{\partial r_1'} - \frac{\partial}{\partial r_1}\right)_{r_1' \to r_1} G_{\omega}^0(\mathbf{r}, \mathbf{l}) G_{\omega}^0(\mathbf{m}, \mathbf{s}) \times \\
&\times \frac{ie\hbar}{mc} A(s_1) \frac{\partial}{\partial s_2} G_{\omega}^0(\mathbf{s}, \mathbf{r}') \times G_{-\omega}^0(\mathbf{m}, \mathbf{l}) - \\
&-\frac{ie\hbar\Delta^2}{m} kT \sum_{\omega} \int \cdots \int d\mathbf{l} d\mathbf{s} d\mathbf{m} \left(\frac{\partial}{\partial r_1'} - \frac{\partial}{\partial r_1}\right)_{r_1' \to r_1} G_{\omega}^0(\mathbf{r}, \mathbf{l}) G_{\omega}^0(\mathbf{m}, \mathbf{r}') G_{-\omega}^0(\mathbf{m}, \mathbf{s}) \times \\
&\times \frac{ie\hbar}{mc} A(s_1) \frac{\partial}{\partial s_2} G_{-\omega}^0(\mathbf{s}, \mathbf{l}).
\end{aligned} \tag{17}$$

Using $\frac{d^2A(r_1)}{dr_1^2} = -\frac{4\pi}{c} j(r_1)$ and equation (15), under the first order approximation, we obtain

$$\begin{aligned}
\frac{d^2\tilde{A}(r_1)}{dr_1^2} = &\frac{4\pi e^2\hbar^2\Delta^2}{m^2c^2} kT \sum_{\omega} \int \cdots \int d\mathbf{l} d\mathbf{s} d\mathbf{m} \left(\frac{\partial}{\partial r_1} - \frac{\partial}{\partial r_1'}\right)_{r_1' \to r_1} G_{\omega}^0(\mathbf{r}, \mathbf{s})(H_0 s_1 + C) \times \\
&\times \frac{\partial}{\partial s_2} G_{\omega}^0(\mathbf{s}, \mathbf{l}) G_{\omega}^0(\mathbf{m}, \mathbf{r}') G_{-\omega}^0(\mathbf{m}, \mathbf{l}) + \\
&+\frac{4\pi e^2\hbar^2\Delta^2}{m^2c^2} kT \sum_{\omega} \int \cdots \int d\mathbf{l} d\mathbf{s} d\mathbf{m} \left(\frac{\partial}{\partial r_1} - \frac{\partial}{\partial r_1'}\right)_{r_1' \to r_1} G_{\omega}^0(\mathbf{r}, \mathbf{l}) G_{\omega}^0(\mathbf{m}, \mathbf{s})(H_0 s_1 + C) \times \\
&\times \frac{\partial}{\partial s_2} G_{\omega}^0(\mathbf{s}, \mathbf{r}') G_{-\omega}^0(\mathbf{m}, \mathbf{l}) + \\
&+\frac{4\pi e^2\hbar^2\Delta^2}{m^2c^2} kT \sum_{\omega} \int \cdots \int d\mathbf{l} d\mathbf{s} d\mathbf{m} \left(\frac{\partial}{\partial r_1} - \frac{\partial}{\partial r_1'}\right)_{r_1' \to r_1} G_{\omega}^0(\mathbf{r}, \mathbf{l}) G_{\omega}^0(\mathbf{m}, \mathbf{r}') G_{-\omega}^0(\mathbf{m}, \mathbf{s}) \times \\
&\times (H_0 s_1 + C) \frac{\partial}{\partial s_2} G_{-\omega}^0(\mathbf{s}, \mathbf{l}).
\end{aligned} \tag{18}$$

Substituting the expression of $G^0_\omega(r,r')$, equation (11), into (18), integrating with respect to the coordinate, and performing a simple combination of terms, we obtain

$$
\begin{aligned}
\frac{d^2\tilde{A}(r_1)}{dr_1^2} = {} & \frac{4\pi e^2\hbar^2\Delta^2(C+H_0d)}{m^2c^2d}\,kT\sum_\omega\iint\frac{dp\,dq}{(2\pi)^2}\sum_k\left\{\frac{2p^2}{(i\omega+\xi)(i\omega-\xi)^3}+\right.\\
& \left.+\frac{p^2}{(i\omega+\xi)^2(i\omega-\xi)^2}\right\}\cos^2 kr_1-\\
& -\frac{2\pi e^2\hbar^2\Delta^2H_0}{m^2c^2d^2}\,kT\sum_\omega\iint\frac{dp\,dq}{(2\pi)^2}\sum_{k_1-k_2=\text{odd}}\left\{\frac{2p^2}{(i\omega-\xi_1)^2(i\omega+\xi_1)(i\omega-\xi_2)}+\right.\\
& \left.+\frac{p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)}\right\}\frac{\cos(k_1-k_2)r_1}{(k_1-k_2)^2}-\\
& -\frac{2\pi e^2\hbar^2\Delta^2H_0}{m^2c^2d^2}\,kT\sum_\omega\iint\frac{dp\,dq}{(2\pi)^2}\sum_{k_1-k_2=\text{odd}}\left\{\frac{2p^2}{(i\omega-\xi_1)^2(i\omega+\xi_1)(i\omega-\xi_2)}+\right.\\
& \left.+\frac{p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)}\right\}\frac{\cos(k_1-k_2)r_1}{(k_1+k_2)^2}.
\end{aligned}
\tag{19}
$$

where $\xi_1 = \frac{\hbar^2}{2m}(k_1^2+p^2+q^2)-\mu$, $\xi_2 = \frac{\hbar^2}{2m}(k_2^2+p^2+q^2)-\mu$; the added /880 condition under the summation sign, $k_1 - k_2$ = odd is the simplified notation of $\frac{2d}{\pi}(k_1-k_2)$ = odd number. Summation is performed over all the positive and negative quantum numbers.

Integrate both sides of equation (19) from 0 to r_1 and select the integration constant C such that $\frac{d\tilde{A}(r_1)}{dr_1}$ obtained satisfies the boundary condition (16). Using this method, we find that

$$C = -H_0d. \tag{20}$$

From equation (19) with the use of (20), we find that

$$\tilde{A}(r_1) = I_1 + I_2, \tag{21}$$

where

$$
\begin{aligned}
I_1 = {} & \frac{2\pi e^2\hbar^2\Delta^2H_0}{m^2c^2d^2}\,kT\sum_\omega\iint\frac{dp\,dq}{(2\pi)^2}\sum_{k_1-k_2=\text{odd}}\left\{\frac{2p^2}{(i\omega-\xi_1)^2(i\omega+\xi_1)(i\omega-\xi_2)}+\right.\\
& \left.+\frac{p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)}\right\}\frac{\cos(k_1-k_2)r_1}{(k_1-k_2)^4},
\end{aligned}
\tag{21a}
$$

$$
\begin{aligned}
I_2 = {} & \frac{2\pi e^2\hbar^2\Delta^2H_0}{m^2c^2d^2}\,kT\sum_\omega\iint\frac{dp\,dq}{(2\pi)^2}\sum_{k_1-k_2=\text{odd}}\left\{\frac{2p^2}{(i\omega-\xi_1)^2(i\omega+\xi_1)(i\omega-\xi_2)}+\right.\\
& \left.+\frac{p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)}\right\}\frac{\cos(k_1-k_2)r_1}{(k_1-k_2)^2(k_1+k_2)^2}.
\end{aligned}
\tag{21b}
$$

First, calculate integral I_1. Let $\nu = k_2 - k_1$. Summation can be considered over k_1 and ν. In general, for a superconducting film which is not too thin ($2d >>>> 10^{-7}$ cm), there is always $\frac{1}{p_0 d} << 1$; therefore, the summation over k, can be converted to an integral. Transform the integral on $dpdqdk_1$, into the spherical coordinate; then it is very easy to prove that the contribution of the third term in $\xi_2 = \xi_1 + \hbar v_0 \nu \cos\theta + \frac{\hbar^2\nu^2}{2m}$ to I_1 can be neglected (here $v_0 = \frac{\hbar p_0}{m}$ is the speed of electrons at the Fermi surface of a normal metal). Therefore, (21a) can be written as

$$I_1 = \frac{e^2\Delta^2 H_0 p_0^3}{\pi m c^2 d} kT \sum_{\omega} \sum_{\nu} \frac{\cos \nu r_1}{\nu^4} \int_{-\infty}^{\infty} d\xi \int_{-1}^{1} dx (1 - x^2) \times$$
$$\times \left\{ \frac{2}{(i\omega - \xi_1)^2 (i\omega + \xi_1)(i\omega - \xi_1 - \hbar v_0 \nu x)} + \right.$$
$$\left. + \frac{1}{(i\omega - \xi_1)(i\omega + \xi_1)(i\omega - \xi_1 - \hbar v_0 \nu x)(i\omega + \xi_1 + \hbar v_0 \nu x)} \right\}.$$

Integrate with respect to ξ:

$$I_1 = \frac{8e^2\Delta^2 H_0 p_0^3}{mc^2 d} kT \sum_{\omega > 0} \frac{1}{\omega} \sum_{\nu} \frac{\cos \nu r_1}{\nu^4} \frac{1}{\hbar^2 v_0^2 \nu^2} \int_{-1}^{1} dx \frac{1 - x^2}{x^2 + \frac{4\omega^2}{\hbar^2 v_0^2 \nu^2}}; \tag{22}$$

Then integrate over x. We then find

$$I_1 = \frac{2}{3} \frac{\Delta^2}{\Delta_0^2(T)} \frac{d^2}{\delta_0^2(T)} \frac{d}{\xi_0} H_0 d \sum_{l=0}^{\infty} \frac{\cos (2l+1)\frac{\pi r_1}{2d}}{(2l+1)^3} \Phi_0\left(\frac{\sigma}{2l+1}\right), \tag{23}$$

where the functions are as follows:

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$$\Phi_0(\eta) = \frac{16}{\pi^3} [G(\eta) - \eta F(\eta)], \tag{23a}$$

$$G(\eta) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \operatorname{tg}^{-1} \frac{1}{(2n+1)\eta}, \tag{23b}$$

$$F(\eta) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \left[1 - (2n+1)\eta \operatorname{tg}^{-1} \frac{1}{(2n+1)\eta}\right], \tag{23c}$$

$$\sigma = 0.728 \frac{d}{\xi_0}, \tag{23d}$$

$\Delta_0(T)$ and $\delta_0(T)$, respectively, express the energy gap and the penetration depth of a weak field of an infinite superconductor, $\xi_0 = 0.182 \frac{\hbar v_0}{kT_c}$.

As to integral I_2, it can be estimated as follows. From (21), it can be seen that the ratio of the two integrands in I_1 and I_2 is

$$\frac{(k_1 + k_2)^2}{(k_1 - k_2)^2} = \frac{(2k_1 + \nu)^2}{\nu^2} \approx \left(1 + \frac{2p_0 x}{\nu}\right)^2 \sim (1 + 2p_0 l x)^2$$

($x = \cos\theta$). From the above calculation of I_1, we can see that when $d \ll \xi_0$, the main contribution to the integral comes from the region $x \sim \frac{2\omega}{\hbar v_0 \nu} \sim 0.36 \frac{2d}{\xi_0}$, and when $d \gg \xi_0$, $x \sim O(1)$. Therefore,

when $d \ll \xi_0$, $$\frac{(k_1 + k_2)^2}{(k_1 - k_2)^2} \sim \left[1 + 0.36 p_0 (2d) \frac{2d}{\xi_0}\right]^2,$$

when $d \gg \xi_0$, $$\frac{(k_1 + k_2)^2}{(k_1 - k_2)^2} \sim (p_0 2d)^2.$$

Therefore, when $d \gg d^*$ the integral I_2 can be neglected. Here $2d^* = \left(\frac{1}{0.36} \frac{\xi_0}{p_0}\right)^{1/2}$. Taking Sn as an example, $\xi_0 = 2.3 \times 10^{-5}$ cm, $v_0 = 0.65 \times 10^8$ cm/sec (Ref. 5), we have $2d^* \sim 10^{-6}$ cm.

In this paper and (Ref. 4), we shall neglect I_2 and similar integrals. This is allowed for relatively thick films ($d \gg d^*$). For thinner films ($d \lesssim d^*$), the contribution of I_2, compared to I_1, can no longer be neglected. But we shall not consider this case in this paper.

Summarizing the above calculation, for a superconducting film with $d^* \ll d < \delta_0(T)$, we obtain

$$A(r_1) = H_0(r_1 - d) + \tilde{A}(r_1), \tag{24}$$

and

$$\tilde{A}(r_1) = \frac{2}{3}\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}\frac{d}{\xi_0}H_0 d\sum_{l=0}^{\infty}\frac{\cos(2l+1)\frac{\pi r_1}{2d}}{(2l+1)^3}\Phi_0\left(\frac{\sigma}{2l+1}\right). \qquad (24a)$$

From the definition of Φ_0 (23a)-(23c), it is not hard to prove:

when $\eta \gg 1$, $\qquad \Phi_0(\eta) \approx \frac{28}{3\pi^3}\zeta(3)\frac{1}{\eta}$,

when $\eta \ll 1$, $\qquad \Phi_0(\eta) \approx 1$.

Since in (24a), almost all contributions come from the terms with smallest ℓ, by using the assymptotic expression, we obtain

for $d \ll \xi_0$, $$\tilde{A}(r_1) = \frac{2}{3}\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}\frac{d}{\xi_0}(H_0 d)\sum_{l=0}^{\infty}\frac{\cos(2l+1)\frac{\pi r_1}{2d}}{(2l+1)^3}, \qquad (25a)$$

and for $d \gg \xi_0$; $$\tilde{A}(r_1) = \frac{1}{3}\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}(H_0 d)\left(1 - \frac{2}{3}\frac{r_1^2}{d^2} + \frac{1}{2}\frac{r_1^3}{d^3}\right). \qquad (25b)$$

(25b) is the same as the result given by the GL theory (Ref. 2, 7).

From equation (24), the magnetic field inside the superconduc- /882 ing film can be obtained instantly

$$H(r_1) = H_0\left\{1 - \frac{\pi}{3}\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}\frac{d}{\xi_0}\sum_{l=0}^{\infty}\frac{\sin\frac{\pi r_1}{2d}}{(2l+1)^4}\Phi_0\left(\frac{\sigma}{2l+1}\right)\right\}. \qquad (26)$$

From equation (24), we can see that $\tilde{A}(r_1)$ is proportional to $\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}$. Therefore, it really is a small quantity. If, furthermore, we substitute (24) and (24a) into (17), we can obtain the correction term proportional to $\frac{\Delta^4}{\Delta_0^4(T)}\frac{d^4}{\delta_0^4(T)}$ by a similar calculation.

V. THE MAGNETIC MOMENT OF A SUPERCONDUCTING FILM

In a magnetic field, the magnetic moment of a superconducting film can generally be expressed as (Ref. 2):

$$\mu = \int_0^{2d}\frac{H(r_1) - H_0}{4\pi}dr_1 = \frac{1}{2\pi}(A(2d) - H_0 d).$$

Using equation (24), we obtain

$$\mu = -\frac{1}{3\pi}\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}\frac{d}{\xi_0}(H_0 d)\sum_{l=0}^{\infty}\frac{\Phi_0\left(\frac{\sigma}{2l+1}\right)}{(2l+1)^5}. \tag{27}$$

It is not difficult to see that

when $d \ll \xi_0$, $$\mu = -\frac{1}{3\pi}\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}\frac{d}{\xi_0}(H_0 d); \tag{28a}$$

and when $d \gg \xi_0$ $$\mu = -\frac{1}{6\pi}\frac{\Delta^2}{\Delta_0^2(T)}\frac{d^2}{\delta_0^2(T)}(H_0 d). \tag{28b}$$

(28b) agrees with the result of the GL theory, but is very much different from (28a). Therefore, it can be seen that the dependence relation of the magnetic moment, μ, and d is different for a thin film and a thick film. The difference between the two reflects the importance of the non-local nature in a thin film.

VI. THE COMPENSATION EQUATION AND THE ENERGY GAP

Under the approximation of constant energy gap, the compensation equation (7) can be written as

$$\tilde{K}\Delta^2 = 2d - K, \tag{29}$$

where

$$\tilde{K} = -|g|kT\sum_{\omega}\int\cdots\int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1\tilde{G}_{\omega}^0(\mathbf{l},\mathbf{m})\tilde{G}_{\omega}^0(\mathbf{s},\mathbf{r})\tilde{G}_{-\omega}^0(\mathbf{s},\mathbf{m})\tilde{G}_{-\omega}^0(\mathbf{l},\mathbf{r}), \tag{30a}$$

$$K = |g|kT\sum_{\omega}\iint d\mathbf{l}dr_1\tilde{G}_{\omega}^0(\mathbf{l},\mathbf{r})\tilde{G}_{-\omega}^0(\mathbf{l},\mathbf{r}). \tag{30b}$$

Substituting (12) for the $\tilde{G}_{\omega}^0(1,r)$ in (30b), and retaining up to the square term of the vector potential, we have

$$K = K_0 + K_2, \tag{31}$$

where

$$K_0 = |g|kT\sum_{\omega}\iint d\mathbf{l}dr_1 G_{\omega}^0(\mathbf{l},\mathbf{r})G_{-\omega}^0(\mathbf{l},\mathbf{r}), \tag{32a}$$

$$K_2 = 2|g|kT \sum_\omega \int \cdots \int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1 G^0_\omega(\mathbf{l}, \mathbf{r}) G^0_{-\omega}(\mathbf{l}, \mathbf{s}) \frac{ie\hbar}{mc} A(s_1) \frac{\partial}{\partial s_2} G^0_{-\omega}(\mathbf{s}, \mathbf{m}) \times$$ /883

$$\times \frac{ie\hbar}{mc} A(m_1) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{r}) +$$

$$+ |g|kT \sum_\omega \int \cdots \int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1 G^0_\omega(\mathbf{l}, \mathbf{s}) \frac{ie\hbar}{mc} A(s_1) \frac{\partial}{\partial s_2} G^0_\omega(\mathbf{s}, \mathbf{r}) G^0_{-\omega}(\mathbf{l}, \mathbf{m}) \times \tag{32b}$$

$$\times \frac{ie\hbar}{mc} A(m_1) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{r}).$$

In (31), the terms with odd orders of the vector potential are zero, and are not written out. Substituting (24) for the vector potential,in (32b), we can write K_2 as follows, accurate to $\frac{\Delta^2}{\Delta_0^2(T)}\ \frac{\delta^2}{\delta_0^2(T)}$

$$K_2 = K_2^{(0)} + K_2^{(1)},$$

where

$$K_2^{(0)} = 2|g|kT \sum_\omega \int \cdots \int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1 G^0_\omega(\mathbf{l}, \mathbf{r}) G^0_{-\omega}(\mathbf{l}, \mathbf{s}) \frac{ie\hbar}{mc} H_0(s_1 - d) \frac{\partial}{\partial s_2} G^0_{-\omega}(\mathbf{s}, \mathbf{m}) \times$$

$$\times \frac{ie\hbar}{mc} H_0(m_1 - d) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{r}) +$$

$$+ |g|kT \sum_\omega \int \cdots \int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1 G^0_\omega(\mathbf{l}, \mathbf{s}) \frac{ie\hbar}{mc} H_0(s_1 - d) \frac{\partial}{\partial s_2} G^0_\omega(\mathbf{s}, \mathbf{r}) \times$$

$$\times G^0_{-\omega}(\mathbf{l}, \mathbf{m}) \frac{ie\hbar}{mc} H_0(m_1 - d) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{r}), \tag{32c}$$

$$K_2^{(1)} = 4|g|kT \sum_\omega \int \cdots \int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1 G^0_\omega(\mathbf{l}, \mathbf{r}) G^0_{-\omega}(\mathbf{l}, \mathbf{s}) \frac{ie\hbar}{mc} H_0(s_1 - d) \frac{\partial}{\partial s_2} G^0_{-\omega}(\mathbf{s}, \mathbf{m}) \times$$

$$\times \frac{ie\hbar}{mc} \tilde{A}(m_1) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{r}) +$$

$$+ 2|g|kT \sum_\omega \int \cdots \int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1 G^0_\omega(\mathbf{l}, \mathbf{s}) \frac{ie\hbar}{mc} \tilde{A}(s_1) \frac{\partial}{\partial s_2} G^0_\omega(\mathbf{s}, \mathbf{r}) G^0_{-\omega}(\mathbf{l}, \mathbf{m}) \times$$

$$\times \frac{ie\hbar}{mc} H_0(m_1 - d) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{r}). \tag{32d}$$

Only the lowest order term is kept for $\tilde{K}$:

$$\tilde{K} = -|g|kT \sum_\omega \int \cdots \int d\mathbf{l}d\mathbf{s}d\mathbf{m}dr_1 G^0_\omega(\mathbf{l}, \mathbf{m}) G^0_\omega(\mathbf{s}, \mathbf{r}) G^0_{-\omega}(\mathbf{s}, \mathbf{m}) G^0_{-\omega}(\mathbf{l}, \mathbf{r}). \tag{33}$$

<u>Calculation of $K_2^{(1)}$</u>. First integrate with respect to r_1, using (11) and (24a); then combine the terms to obtain

$$K_2^{(1)} = |g| \frac{\pi e^4 \Delta^2 H_0^2 \hbar p_0^3}{m^3 c^4 v_0} \sum_\omega \iint \frac{dp\,dq}{(2\pi)^2} \frac{1}{(2d)^2} \sum_{\substack{k_1=-\infty \\ k_1-k_2=\overline{}}}^{\infty} \sum_{k_2=-\infty}^{\infty} \left\{ \frac{4p^2}{(i\omega-\xi_1)(i\omega+\xi_1)^2(i\omega+\xi_2)} + \right.$$

$$\left. + \frac{2p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)} \right\} \frac{\Phi_0\left(\frac{\pi\sigma}{2d|k_1-k_2|}\right)}{|k_1-k_2|^7} +$$

$$+ |g| \frac{\pi e^4 \Delta^2 H_0^2 \hbar p_0^3}{m^3 c^4 v_0} \sum_\omega \iint \frac{dp\,dq}{(2\pi)^2} \frac{1}{(2d)^2} \sum_{\substack{k_1=-\infty \\ k_1-k_2=\overline{}}}^{\infty} \sum_{k_2=-\infty}^{\infty} \left\{ \frac{4p^2}{(i\omega-\xi_1)(i\omega+\xi_1)^2(i\omega+\xi_2)} + \right. \quad (34)$$

$$\left. + \frac{2p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)} \right\} \frac{\Phi_0\left(\frac{\pi\sigma}{2d|k_1-k_2|}\right)}{|k_1-k_2|^5|k_1+k_2|^2},$$

where the function $\Phi_0(\eta)$ is defined by (23a)-(23c). When d >> d*, /884 the second integral in (34) can be neglected and the first integral can be obtained by the method of the fourth section.

$$K_2^{(1)} = |g| N(0) \frac{255\zeta(8) e^4 \Delta^2 p_0^3 H_0^2 d^7}{2\pi^5 m c^4 \hbar^2 (kT)^2} \Phi_2(\sigma), \quad (35)$$

where

$$\Phi_2(\sigma) = \frac{256}{255\zeta(8)} \sum_{l=0}^{\infty} \frac{\Phi_0^2\left(\frac{\sigma}{2l+1}\right)}{(2l+1)^8}. \quad (35a)$$

Similarly, when d >> d*, we obtain

$$K_2^{(0)} = -|g| N(0) \frac{31\zeta(5) p_0 H_0^2 d^4}{2\pi^3 m c^2 kT} \Phi_1(\sigma), \quad (36)$$

where

$$\Phi_1(\sigma) = \frac{32}{31\zeta(5)} \sum_{l=0}^{\infty} \frac{\Phi_0\left(\frac{\sigma}{2l+1}\right)}{(2l+1)^5}. \quad (36a)$$

The two integrals remaining, K_0 and $\tilde{K}$, are very easy to calculate, and we shall only write down the results as follows:

$$K_0 = 2d\left[1 + |g| N(0) \ln \frac{T_c}{T}\right] \approx 2d[1 + gN(0)\Delta t], \quad (37)$$

$$\tilde{K} = -|g| N(0) \frac{7\zeta(3) d}{4\pi^2 (kT)^2}. \quad (38)$$

Substituting the obtained K_0, $K_2^{(0)}$, $K_2^{(1)}$ and $\tilde{K}$ into (29, we can calculate the energy gap of a superconducting film immediately:

$$\frac{\Delta^2}{\Delta_0^2(T)} = \frac{1 - \frac{1}{3}\frac{d^2}{\delta_0^2(T)}\frac{d}{\xi_0}\Phi_1(\sigma)\frac{H_0^2}{H_{CM}^2}}{1 - 0.516\frac{d^4}{\delta_0^4(T)}\frac{d^2}{\xi_0^2}\Phi_2(\sigma)\frac{H_0^2}{H_{CM}^2}}. \tag{39}$$

When d << $\delta_0(T)$, the second term of the denominator can be neglected, and thus

$$\frac{\Delta^2}{\Delta_0^2(T)} = 1 - \frac{1}{3}\frac{d^2}{\delta_0^2(T)}\frac{d}{\xi_0}\Phi_1(\sigma)\frac{H_0^2}{H_{CM}^2}. \tag{40}$$

Using the asymptotic expression of $\Phi_1(\sigma)$ at $\sigma >> 1$ and $\sigma << 1$

$$\Phi_1(\sigma) \approx 1 \qquad (\sigma \ll 1),$$
$$\Phi_1(\sigma) \simeq \frac{28\zeta(3)\pi}{279\zeta(5)\sigma} \qquad (\sigma \gg 1),$$

we get

when d << ξ_0
$$\frac{\Delta^2}{\Delta_0^2(T)} = 1 - \frac{1}{3}\frac{d}{\xi_0}\frac{d^2}{\delta_0^2(T)}\frac{H_0^2}{H_{CM}^2}, \tag{41a}$$
and when d >> ξ_0
$$\frac{\Delta^2}{\Delta_0^2(T)} = 1 - \frac{1}{6}\frac{d^2}{\delta_0^2(T)}\frac{H_0^2}{H_{CM}^2}. \tag{41b}$$

(41b) agrees with the GL theory. Here, we find the same situation as in sections four and five - that is, for a relatively thin film (d < ξ_0), our result is different from that of the GL theory, and for a relatively thick film (d >> ξ_0) the two tend to agree.

VII. THE PHASE TRANSITION OF A SUPERCONDUCTING THIN FILM IN A MAGNETIC FIELD /885

Using the results of the previous section, we can study the phase transition problem of a superconducting thin film in a magnetic field. For this purpose, let us first establish the expression for the free energy.

In a magnetic field, the difference in free energy density of a thin metal film at the superconducting phase and the normal phase can be expressed as (Ref. 3):

$$F_{sH} - F_{nH} = \int_0^{\Delta} d\Delta |\Delta|^2 \frac{\delta \left|\frac{1}{g}\right|}{\delta\Delta}. \tag{42}$$

Under the approximation of a constant energy gap, the compensation equation can be written as

$$2d = |g| kT \sum_{\omega} \iint d\mathbf{l} d r_1 \tilde{G}^0_{\omega}(\mathbf{l}, \mathbf{r}) \tilde{G}^0_{-\omega}(\mathbf{l}, \mathbf{r}) -$$
$$- |g| kT \sum_{\omega} \int \cdots \int d\mathbf{l} d\mathbf{m} d\mathbf{s} d r_1 \tilde{G}^0_{\omega}(\mathbf{l}, \mathbf{m}) \tilde{G}^0_{\omega}(\mathbf{s}, \mathbf{r}) \tilde{G}^0_{-\omega}(\mathbf{s}, \mathbf{m}) \tilde{G}^0_{-\omega}(\mathbf{l}, \mathbf{r}) \Delta^2,$$

Therefore, we have

$$\frac{\delta \left|\frac{1}{g}\right|}{\delta\Delta} = \frac{\tilde{K}}{2|g|d} \Delta, \tag{43}$$

where $\tilde{K}$ is given by (30a). Substituting (43) into (42), we obtain

$$F_{sH} - F_{nH} = \frac{1}{2} |\Delta|^4 \frac{\tilde{K}}{2|g|d}.$$

Applying (29) again, we find the free energy density of a superconducting film in a magnetic field

$$F_{sH} = F_{n0} + \frac{H^2(r_1)}{8\pi} + \frac{1}{2|g|} \left(1 - \frac{1}{2d} K\right) |\Delta|^2, \tag{44}$$

where F_{n0} expresses the free energy density of the thin film at normal phase when H = 0.

The critical magnetic field H_c of a superconducting film is given by the following thermodynamic equation (Ref. 2):

$$\frac{H_C^2}{8\pi} = \frac{H_{CM}^2}{8\pi} - \frac{\sigma}{2d}, \tag{45}$$

where

$$\sigma = \int_0^{2d} \left[\frac{H^2(r_1)}{8\pi} - \frac{1}{4\pi} H_C H(r_1) + \Delta F\right] d r_1, \tag{46}$$

ΔF designates the free energy difference of a superconducting film with magnetic field and without magnetic field. H_{CM} expresses the critical magnetic field of an infinite superconductor. Using (44) and the relation $F_{n0} - F_{s0} = \frac{H^2_{cm}}{8\pi}$, we obtain

$$\Delta F = \frac{1}{2|g|}\left(1 - \frac{1}{2d}K\right)|\Delta|^2 + \frac{H^2_{CM}}{8\pi}. \tag{47}$$

Substituting (47) into (45), we have

$$H_C^2 = -\frac{4\pi}{|g|}\left(1 - \frac{1}{2d}K\right)|\Delta|^2 - \frac{1}{2d}\int_0^{d}[H^2(r_1) - 2H_C H(r_1)]dr_1.$$

Under the situation we are discussing, $H(r_1) = H_c + \frac{d\tilde{A}(r_1)}{dr_1}$. Applying the result calculated in Section IV, we obtain the following relation /886

$$-\frac{4\pi}{|g|}\left(1 - \frac{1}{2d}K\right) = \frac{2295\gamma^2\zeta(8)}{98\pi^4\zeta^2(3)}\,\frac{\Delta^2 d^6 H_C^2}{\Delta_0^4(T)\delta_0^4(T)\xi_0^2}\,\Phi_2(\sigma), \tag{48}$$

where $\gamma = \ln C = 1.78$, K is given by (30b) and (31)-(32d), and only H_0 must be replaced by H_c.

(48) and (39) are simultaneous equations determining the critical magnetic field H_c and the energy gap of a superconducting film at $H_0 = H_c$. It is very easy to see that these two equations allow a solution with $\Delta^2 = 0$ and with H_c determined by the following equation:

$$\frac{1}{2d}K - 1 = 0. \tag{49}$$

K, which is accurate to the H^2 term, has already been calculated in Section VI. Using the result of Section VI, we obtain from (49) the critical magnetic field of the superconducting thin film as (d >> d*)

$$\frac{H_C^2}{H_{CM}^2} = 3\frac{\delta_0^2(T)}{d^2}\frac{\xi_0}{d}\frac{1}{\Phi_1(\sigma)}, \tag{50}$$

The phase transition taking place at this time is a second order one. Using the assymptotic expression of $\Phi_1(\sigma)$ at $\sigma >> 1$ and $\sigma << 1$, we can obtain the two limiting cases of (50):

when $d << \xi_0$ $$\frac{H_C^2}{H_{CM}^2} = 3\frac{\delta_0^2(T)}{d^2}\frac{\xi_0}{d};$$

when $d \gg \xi_0$ $$\frac{H_C^2}{H_{CM}^2} = 6\frac{\delta_0^2(T)}{d^2}.$$

The latter agrees with the GL theory (Ref. 2, 7). Similar to the GL theory, there is a critical thickness d_c. When $d < d_c$, the phase transition is a second order one and when $d > d_c$, the phase transition is of first order. From (39), we know d_c is determined by the following equation:

$$1 - 0.516\frac{d_C^4}{\delta_0^4(T)}\frac{d_C^2}{\xi_0^2}\Phi_2(\sigma)\frac{H_C^2}{H_{CM}^2} = 0.$$

Substituting (50) into the above equation, we obtain the critical thickness d_c:

$$\frac{d_C^2}{\delta_0^2(T)} = \frac{1}{1.548}\frac{\xi_0}{d_C}\frac{\Phi_1(\sigma)}{\Phi_2(\sigma)}. \qquad (51)$$

Using the asymptotic expressions of $\Phi_1(\sigma)$ and $\Phi_2(\sigma)$ at $\sigma \gg 1$, we obtain, if $d_c \gg \xi_0$, the following

$$\frac{d_C^2}{\delta_0^2(T)} = \frac{5}{4}\left(1 + 0.286\frac{\xi_0^2}{d_C^2} + \cdots\right). \qquad (52)$$

Obviously, when the temperature is close enough to T_c, we would have $d_c \gg \xi_0$. Therefore, in the vicinity of T_c, (52) is always valid. Except for unimportant corrections, this criterion agrees with the result of the GL theory.

VIII. CRITICAL CURRENT

Using the method of Sections IV through VII, we can also calculate the critical current. Assume the sample is a cylindrical column, with its radius greater than the difference of the outside and inside diameter. We can approximately treat it as a thin film (Ref. 6). Let R_1 and R_2 be the inside and outside radius, respectively, $R_2 - R_1 = 2d$. Introduce the coordinate $r_1 = R - R_1$ ($0 \leqslant r_1 \leqslant 2d$. Let the total current through the sample be J, and then the boundary conditions are at $r_1 = 0$, $\quad H_J = 0;$

at $r_1 = 2d$, $\quad H_J = \frac{2J}{cR_2}.$ (53)

The relationship between the current $i(r_1)$ and the vector potential $A(r_1)$ is still given by (17). Using the method of Section IV, we assume /887

$$A(r_1) = \frac{H_J r_1^2}{4d} + C + \tilde{A}(r_1), \tag{54}$$

where C is a constant, and the boundary condition which $\tilde{A}(r_1)$ satisfies is still (16). Substituting (54) into (17), under the first order approximation, it can be written as

$$\begin{aligned}\frac{H_J^2}{2d} + \frac{d^2\tilde{A}(r_1)}{dr_1^2} = {} & \frac{4\pi e^2\Delta^2\hbar^2}{m^2c^2} kT \sum_\omega \iint \frac{dpdq}{(2\pi)^2} \sum_{k>0} \Big[\frac{4p^2}{(i\omega-\xi)^3(i\omega+\xi)} + \\ & + \frac{2p^2}{(i\omega-\xi)^2(i\omega+\xi)^2}\Big]\left(\frac{H_J d}{3} + C\right)\varphi_k^2(r_1) - \\ & - \frac{2\pi e^2\Delta^2\hbar^2}{m^2c^2} kT \sum_\omega \iint \frac{dpdq}{(2\pi)^2} \sum_{\substack{k_1 \\ k_1-k_2=\alpha(\neq 0)}} \sum_{k_2} \Big[\frac{2p^2}{(i\omega-\xi_1)(i\omega-\xi_2)^2(i\omega+\xi_2)} + \\ & + \frac{p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)}\Big]\frac{1}{(k_1-k_2)^2}\varphi_{k_1}(r_1)\varphi_{k_2}(r_1) + \\ & + \frac{2\pi e^2\Delta^2\hbar^2}{m^2c^2} kT \sum_\omega \iint \frac{dpdq}{(2\pi)^2} \sum_{\substack{k_1 \\ k_1-k_2=\pi}} \sum_{k_2} \Big[\frac{2p^2}{(i\omega-\xi_1)(i\omega-\xi_2)^2(i\omega+\xi_2)} + \\ & + \frac{p^2}{(i\omega-\xi_1)(i\omega+\xi_1)(i\omega-\xi_2)(i\omega+\xi_2)}\Big]\frac{1}{(k_1-k_2)^2}\varphi_{k_1}(r_1)\varphi_{k_2}(r_1).\end{aligned} \tag{55}$$

In (55) we neglected the following integral

$$\frac{4\pi e^2\Delta^2\hbar^2}{m^2c^2} kT \sum_\omega \iint \frac{dpdq}{(2\pi)^2} \sum_{k>0} \Big[\frac{4p^2}{(i\omega-\xi)^3(i\omega+\xi)} + \frac{2p^2}{(i\omega-\xi)^2(i\omega+\xi)^2}\Big]\frac{1}{2k^2}\varphi_k^2(r_1).$$

Its contribution is only $\frac{1}{p_0^2 d^2}$ of the first integral in (55) and can be neglected under the situation we are considering. Using the method of Section IV, we can determine the constant C and obtain $\tilde{A}(r_1)$:

$$C = \frac{1}{2}\frac{H_J\delta_0^2(T)}{d}\frac{\Delta_0^2}{\Delta^2} - \frac{H_J d}{3}, \tag{56}$$

$$\tilde{A}(r_1) = 0.82\frac{\Delta^2}{\Delta_0^2(T)}\frac{1}{\delta_0^2(T)}\frac{d}{\xi_0}H_Jd\left(\frac{2d}{\pi}\right)^2\sum_{l=0}^{\infty}\frac{\cos(2l+1)\frac{\pi r_1}{2d}}{(2l+1)^3}\Phi_0\left(\frac{\sigma}{2l+1}\right) - \qquad (57)$$

$$-0.82\frac{\Delta^2}{\Delta_0^2(T)}\frac{1}{\delta_0^2(T)}\frac{d}{\xi_0}H_Jd\left(\frac{2d}{\pi}\right)^2\sum_{l=1}^{\infty}\frac{\cos 2l\frac{\pi r_1}{2d}}{(2l)^3}\Phi_0\left(\frac{\sigma}{2l}\right).$$

Now let us calculate the energy gap. The compensation equation is

$$\tilde{K}\Delta^2 = 2d - K_0 - K_2, \qquad (31)$$

where $\tilde{K}$ and K_0 are still given by (40) and (41). K_2 is given by (34b), but $A(r_1)$ is defined by (54). The calculation shows that the critical current is mainly determined by the first two terms in (54), and the contribution of $\tilde{A}(r_1)$ can be neglected. Therefore,

$$K_2 = 2|g|kT\sum_{\omega}\int\cdots\int d\mathbf{l}d\mathbf{s}d\mathbf{m}d\mathbf{r}_1 G_\omega^0(\mathbf{l},\mathbf{r})G_{-\omega}^0(\mathbf{l},\mathbf{s})\frac{ie\hbar}{mc}\left(\frac{H_Js_1^2}{4d}+C\right)\frac{\partial}{\partial s_2}G_{-\omega}^0(\mathbf{s},\mathbf{m})\times$$

$$\times\frac{ie\hbar}{mc}\left(\frac{H_Jm_1^2}{4d}+C\right)\frac{\partial}{\partial m_2}G_{-\omega}^0(\mathbf{m},\mathbf{r}) +$$

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$$+|g|kT\sum_{\omega}\int\cdots\int d\mathbf{l}d\mathbf{s}d\mathbf{m}d\mathbf{r}_1 G_\omega^0(\mathbf{l},\mathbf{s})\frac{ie\hbar}{mc}\left(\frac{H_Js_1^2}{4d}+C\right)\frac{\partial}{\partial s_2}G_{-\omega}^0(\mathbf{s},\mathbf{r})G_{-\omega}^0(\mathbf{l},\mathbf{m})\times \qquad (58)$$

$$\times\frac{ie\hbar}{mc}\left(\frac{H_Jm_1^2}{4d}+C\right)\frac{\partial}{\partial m_2}G_{-\omega}^0(\mathbf{m},\mathbf{r}).$$

After a calculation similar to that of Sections IV to VII, we obtain

$$K_2 = -\frac{7\zeta(3)e^2\hbar^2p_0^2d}{6\pi^2mc^2(kT)^2}|g|N(0)\left(\frac{H_Jd}{3}+C\right) - \frac{4\zeta(5)e^2p_0H_J^2d^4}{\pi^3mc^2kT}|g|N(0). \qquad (59)$$

Thus, we obtain

$$\Delta^2 = \Delta_0^2 - \frac{2e^2\hbar^2p_0^2}{3m^2c^2}\left(\frac{H_Jd}{3}+C\right)^2 - \frac{4\zeta(5)e^2p_0kTH_J^2d^3}{7\zeta(3)\pi^2mc^2}. \qquad (60)$$

The critical current is determined by the equation $\frac{dH_J}{d\Delta^2} = 0$ (Ref. 6). From (60), we obtain

$$\frac{H_{JC}}{H_{CM}} = \frac{2d}{\delta_0(T)}\frac{\Delta_c^3}{\Delta_0^3(T)}, \qquad (61)$$

where H_{J_c} is the magnetic field generated on the film surface by the critical current J_c. The relationship between H_{J_c} and J_c is given by (53). Δ_c expresses the value of the energy gap when $H_J = H_{J_c}$. From (60) and (61), we know

$$\Delta_c^2 = \frac{2}{3}\Delta_0^2(T). \tag{62}$$

Substituting (62) into (61), we obtain

$$\frac{H_{Jc}}{H_{CM}} = \frac{2\sqrt{2}}{3\sqrt{3}}\frac{2d}{\delta_0(T)}. \tag{63}$$

This is the result we desired. Comparing (50) with (63), we have

$$H_{Jc}H_c = \frac{4\sqrt{2}}{3}\left[\frac{\xi_0}{\Phi_1(\sigma)d}\right]^{\frac{1}{2}} H_{CM}^2. \tag{64}$$

It is easy to see that

when $d \ll \xi_0$, $$H_{Jc}H_c = \frac{4\sqrt{2}}{3}\left(\frac{\xi_0}{d}\right)^{\frac{1}{2}} H_{CM}^2; \tag{65a}$$

and when $d \gg \xi_0$ $$H_{Jc}H_c = \frac{8}{3} H_{CM}^2. \tag{65b}$$

The expression for H_{J_c} (63) obtained in this section agrees with the GL theory (Ref. 6). H_{J_c} of the superconducting thin film is the only quantity which agrees with the GL theory, no matter what the thickness is. From our derivation, it is not difficult to understand that the non-local effect is not important to this physical quantity.

IX. CONCLUSION

This article studied in detail the effect of the non-local character, induced by the dimensions of the sample, on the behavior of superconducting thin films in a magnetic field. A theory of superconductivity of a thin metal film in a magnetic field is given (in a temperature range near T_c) and expressions for the magnetic moment, energy gap, critical magnetic field, and critical current of a superconducting thin film ($d^* \ll d < \delta_0(T)$) are obtained.

The results of this paper show that the GL theory is applicable to a film with enough thickness ($d \gg \xi_0$), but as the film thickness decreases - especially when $d \ll \xi_0$ - there is considerable disagreement between the result of this paper and that of GL theory (except for the critical current). Taking the magnetic moment as an example, the theory of this paper expects that when $d \ll \xi_0$, it is proportional to $\frac{d^2}{\delta_0^2(T)} \frac{d}{\xi_0}$. According to the GL theory, it is proportional to $\frac{d^2}{\delta_0^2(T)}$. /889

In (Ref. 4), we compared the theoretical formula of the critical magnetic moment with the experiment and found very good agreement.

When carrying out this research we had many valuable discussions with professor Wong Tzu-si and comrade Yu Lu. We would like to express our gratitude.

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